TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 211, 1975

MUTUAL EXISTENCE OF PRODUCT INTEGRALS IN NORMED RINGS

BY

JON C. HELTON[1]

ABSTRACT. Definitions and integrals are of the subdivision-refinement type, and functions are from $R \times R$ to N, where R denotes the set of real numbers and N denotes a ring which has a multiplicative identity element represented by 1 and a norm $|\cdot|$ with respect to which N is complete and $|1| = 1$. If G is a function from $R \times R$ to N, then $G \in OM^*$ on $[a, b]$ only if (i) ${}_x\Pi^y(1 + G)$ exists for $a \leq x < y \leq b$ and (ii) if $\epsilon > 0$, then there exists a subdivision D of $[a, b]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D and $0 \leq p < q \leq n$, then

$$\left| {}_{x_p}\Pi^{x_q}(1 + G) - \prod_{i=p+1}^{q}(1 + G_i) \right| < \epsilon;$$

and $G \in OM^\circ$ on $[a, b]$ only if (i) ${}_x\Pi^y(1 + G)$ exists for $a \leq x < y \leq b$ and (ii) the integral $\int_a^b |1 + G - \Pi(1 + G)|$ exists and is zero. Further, $G \in OP^\circ$ on $[a, b]$ only if there exist a-subdivision D of $[a, b]$ and a number B such that, if $\{x_i\}_{i=0}^n$ is a refinement of D and $0 < p \leq q \leq n$, then $|\Pi_{i=p}^q(1 + G_i)| < B$.

If F and G are functions from $R \times R$ to N, $F \in OP^\circ$ on $[a, b]$, each of $\lim_{x,y\to p^+} F(x, y)$ and $\lim_{x,y\to p^-} F(x, y)$ exists and is zero for $p \in [a, b]$, each of $\lim_{x\to p^+} F(p, x)$, $\lim_{x\to p^-} F(x, p)$, $\lim_{x\to p^+} G(p, x)$ and $\lim_{x\to p^-} G(x, p)$ exists for $p \in [a, b]$, and G has bounded variation on $[a, b]$, then any two of the following statements imply the other:

(1) $F + G \in OM^*$ on $[a, b]$, (2) $F \in OM^*$ on $[a, b]$, and (3) $G \in OM^*$ on $[a, b]$.

In addition, with the same restrictions on F and G, any two of the following statements imply the other:

(1) $F + G \in OM^\circ$ on $[a, b]$, (2) $F \in OM^\circ$ on $[a, b]$, and (3) $G \in OM^\circ$ on $[a, b]$.

The results in this paper generalize a theorem contained in a previous paper by the author [Proc. Amer. Math. Soc. 42 (1974), 96–103]. Additional background on product integration can be obtained from a paper by B. W. Helton [Pacific J. Math. 16 (1966), 297–322].

Presented to the Society, January 23, 1975; received by the editors October 4, 1974.
AMS (MOS) subject classifications (1970). Primary 28A25, 26A39.
Key words and phrases. Sum integral, product integral, subdivision-refinement integral, existence, interval function, normed complete ring.
[1] This research was supported in part by a grant from Arizona State University.

All definitions are of the subdivision-refinement type, and functions are from $R \times R$ to N, where R denotes the set of real numbers and N denotes a ring which has a multiplicative identity element represented by 1 and a norm $|\cdot|$ with respect to which N is complete and $|1| = 1$. Functions are assumed to be defined only for elements $\{x, y\}$ of $R \times R$ such that $x < y$.

If G is a function from $R \times R$ to N, then $\int_a^b G$ exists only if there exists an element L of N such that, if $\epsilon > 0$, then there exists a subdivision D of $[a, b]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D, then $|L - \Sigma_{i=1}^n G_i| < \epsilon$, where $G_i = G(x_{i-1}, x_i)$. Similarly, ${}_a\Pi^b(1 + G)$ exists only if there exists an element L of N such that, if $\epsilon > 0$, then there exists a subdivision D of $[a, b]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D, then $|L - \Pi_{i=1}^n (1 + G_i)| < \epsilon$.

The statements that G is bounded on $[a, b]$, $G \in OP^\circ$ on $[a, b]$ and $G \in OB^\circ$ on $[a, b]$ mean there exist a subdivision D of $[a, b]$ and a number B such that, if $\{x_i\}_{i=0}^n$ is a refinement of D, then

(1) $|G_i| < B$ for $1 \leq i \leq n$,

(2) $|\Pi_{i=p}^q (1 + G_i)| < B$ for $1 \leq p \leq q \leq n$, and

(3) $\Sigma_{i=1}^n |G_i| < B$,

respectively.

Let $G(p, p^+)$, $G(p^+, p^+)$, $G(p^-, p)$ and $G(p^-, p^-)$ represent $\lim_{x \to p^+} G(p, x)$, $\lim_{x,y \to p^+} G(x, y)$, $\lim_{x \to p^-} G(x, p)$ and $\lim_{x,y \to p^-} G(x, y)$, respectively. Now, $G \in S_1$ on $[a, b]$ only if $G(p^+, p^+)$ exists and is zero for $a \leq p < b$ and $G(p^-, p^-)$ exists and is zero for $a < p \leq b$; and $G \in S_2$ on $[a, b]$ only if $G(p, p^+)$ exists for $a \leq p < b$ and $G(p^-, p)$ exists for $a < p \leq b$. Further, $G \in OL^\circ$ on $[a, b]$ only if $G(p, p^+)$ and $G(p^+, p^+)$ exist for $a \leq p < b$ and $G(p^-, p)$ and $G(p^-, p^-)$ exist for $a < p \leq b$.

For additional background on product integration, the reader is referred to papers by P. R. Masani [10], J. S. MacNerney [9], B. W. Helton [2] and the author [7].

Suppose F and G are functions on $R \times R$. If $\int_a^b F$ exists and $\int_a^b G$ exists, then it is easily shown that $\int_a^b (F + G)$ exists. However, if ${}_x\Pi^y(1+F)$ and ${}_x\Pi^y(1 + G)$ exist for $a \leq x < y \leq b$, it does not necessarily follow that ${}_x\Pi^y(1 + F + G)$ exists for $a \leq x < y \leq b$. The purpose of this paper is to investigate the existence of such product integrals. In particular, with suitable restrictions on the functions involved, we interrelate the existence of ${}_x\Pi^y(1 + F)$, ${}_x\Pi^y(1 + G)$ and ${}_x\Pi^y(1 + F + G)$. However, before stating our results, we need several additional definitions.

First, $G \in OA^\circ$ on $[a, b]$ only if $\int_a^b G$ exists and $\int_a^b |G - \int G| = 0$. Second, $G \in OM^\circ$ on $[a, b]$ only if ${}_x\Pi^y(1 + G)$ exists for $a \leq x < y \leq b$ and $\int_a^b |1 + G - \Pi(1 + G)| = 0$. Third, $G \in OM^*$ on $[a, b]$ only if (1) ${}_x\Pi^y(1 + G)$

exists for $a \leq x < y \leq b$, and (2) if $\epsilon > 0$, then there exists a subdivision D of $[a, b]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D and $0 \leq p < q \leq n$, then

$$\left| {}_{x_p}\prod^{x_q}(1 + G) - \prod_{i=p+1}^{q}(1 + G_i) \right| < \epsilon.$$

We now state the main results of this paper.

Theorem 1. *If F and G are functions from $R \times R$ to N, F is in OP° and $S_1 \cap S_2$ on $[a, b]$ and G is in OB° and S_2 on $[a, b]$, then any two of the following statements imply the other:*

(1) $F + G \in OM^*$ *on* $[a, b]$,

(2) $F \in OM^*$ *on* $[a, b]$, *and*

(3) $G \in OM^*$ *on* $[a, b]$.

Theorem 2. *If F and G are functions from $R \times R$ to N, F is in OP° and $S_1 \cap S_2$ on $[a, b]$ and G is in OB° and S_2 on $[a, b]$, then any two of the following statements imply the other:*

(1) $F + G \in OM^\circ$ *on* $[a, b]$,

(2) $F \in OM^\circ$ *on* $[a, b]$, *and*

(3) $G \in OM^\circ$ *on* $[a, b]$.

Theorems 1 and 2 are not the same. A function can belong to OM^* on $[a, b]$ without belonging to OM° on $[a, b]$. For example, if $G \in OB^\circ$ on $[a, b]$ and ${}_x\Pi^y(1 + G)$ exists for $a \leq x < y \leq b$, then $G \in OM^*$ on $[a, b]$ [7, Theorem 1]; but, it is possible to construct a function G such that $G \in OB^\circ$ on $[a, b]$, ${}_x\Pi^y(1 + G)$ exists for $a \leq x < y \leq b$ and $G \notin OM^\circ$ on $[a, b]$ [4, pp. 153–154]. However, if G is in OM° and OP° on $[a, b]$, then $G \in OM^*$ on $[a, b]$.

Theorem 2 is proved for functions from $R \times R$ to R in a previous paper by the author [6, Theorem 1, p. 101]. However, that proof relies heavily on the commutativity of R and thus is not the same as the proof presented in this paper. In this presentation, the lack of commutativity is handled by using a series representation for products.

The classes OM^* and OM° are not as restricted as may initially appear. As noted before, if $G \in OB^\circ$ on $[a, b]$ and ${}_x\Pi^y(1 + G)$ exists for $a \leq x < y \leq b$, then $G \in OM^*$ on $[a, b]$ [7, Theorem 1]. For another example, suppose

$$F(x, y) = \begin{bmatrix} 0 & 0 \\ h(y) - h(x) & 0 \end{bmatrix}$$

for $a \leq x < y \leq b$, where h is a quasi-continuous function from R to N. Then, with a suitable norm, F is in OP°, OM° and $S_1 \cap S_2$ on $[a, b]$. Thus, F

satisfies the hypotheses of Theorems 1 and 2; however, it does not necessarially follow that $F \in OB^\circ$ on $[a, b]$. With Theorems 1 and 2 and functions such as F, it is possible to construct many functions in OM^* and OM°. A fundamental correspondence exists between sum and product integrals. In particular, if $G \in OB^\circ$ on $[a, b]$, then $\int_a^b G$ exists if and only if ${}_x\Pi^y(1+G)$ exists for $a \le x < y \le b$ [7, Theorem 4], and $G \in OA^\circ$ on $[a, b]$ if and only if $G \in OM^\circ$ on $[a, b]$ [2, Theorem 3.4, p. 301]. If G is a function from $R \times R$ to R, then $G \in OA^\circ$ on $[a, b]$ [8, p. 669]. Further, there exist other systems such that the existence of $\int_a^b G$ is sufficient to imply that $G \in OA^\circ$ on $[a, b]$ [1, Theorem 2, p. 155], [2, Theorem 4.1, p. 304]. Thus, with the preceding results, many functions in OM^* and OM° can be obtained. In addition, if $H \in OL^\circ$ on $[a, b]$ and $G \in OB^\circ$ on $[a, b]$, then ${}_x\Pi^y(1+HG)$ exists for $a \le x < y \le b$ if ${}_x\Pi^y(1+G)$ exists for $a \le x < y \le b$ [7, Theorem 5], and $HG \in OM^\circ$ on $[a, b]$ if $G \in OM^\circ$ on $[a, b]$ [3, Theorem 2, p. 494]. Therefore, there exist many functions to which the theorems of this paper apply.

We now establish Theorem 1. Several lemmas are needed.

Lemma 1. *If H and G are functions from $R \times R$ to N, $H \in OL^\circ$ on $[a, b]$, $G \in OB^\circ$ on $[a, b]$ and either $\int_a^b G$ exists or ${}_x\Pi^y(1+G)$ exists for $a \le x < y \le b$, then $\int_a^b HG$ and $\int_a^b GH$ exist and ${}_x\Pi^y(1+HG)$ and ${}_x\Pi^y(1+GH)$ exist for $a \le x < y \le b$ [7, Theorem 5].*

Lemma 2. *If f is a function from R to R such that $(LR)\int_a^b(-df)f^{n-i}f^i$ exists for $i = 0, 1, \ldots, n$, then*

$$\sum_{i=0}^{n}(LR)\int_a^b(-df)f^{n-i}f^i = f^{n+1}(a) - f^{n+1}(b).$$

Proof. This result follows by applying the identity

$$(r-s)\sum_{i=0}^{n} r^{n-i}s^i = r^{n+1} - s^{n+1}$$

to the approximating sums of the integrals involved.

Lemma 3. *If $\{F_i\}_{i=m}^n$ and $\{G_i\}_{i=m}^n$ are sequences of elements of N, then*

$$\prod_{i=m}^{n}(1+F_i+G_i) = \sum_{i=0}^{n+1-m} S_{imn},$$

where

$$S_{0pn} = \begin{cases} \Pi_{j=p}^n(1+F_j) & \text{if } 0 < p \le n, \\ 1 & \text{if } p > n, \end{cases}$$

and

$$S_{ipn} = \begin{cases} \Sigma_{j=p}^{n} [\Pi_{k=p}^{j-1}(1 + F_k)] G_j S_{i-1,j+1,n} & \text{if } 0 < p \leq n, \\ 0 & \text{if } p > n \end{cases}$$

for $i = 1, 2, \ldots$.

Proof. This lemma can be established by induction.

Lemma 4. *If* F *and* G *are functions from* $R \times R$ *to* N, $F \in OP^\circ$ *on* $[a, b]$ *and* $G \in OB^\circ$ *on* $[a, b]$, *then there exist a subdivision* D *of* $[a, b]$, *a number* B *and a positive nondecreasing function* g *defined on* $[a, b]$ *such that, if* $\{x_i\}_{i=0}^{n}$ *is a refinement of* D, j *is a nonnegative integer and* $0 < p \leq q \leq n$, *then*

$$(\dagger) \qquad |S_{jpq}| \leq B^{j+1}[g(x_q) - g(x_{p-1})]^j/j!,$$

where S_{jpq} *is defined in Lemma* 3.

Proof. Since $F \in OP^\circ$ on $[a, b]$ and $G \in OB^\circ$ on $[a, b]$, there exist a subdivision D of $[a, b]$ and a number B such that, if $\{x_i\}_{i=0}^{n}$ is a refinement of D, then

(1) $|\Pi_{i=p+1}^{q}(1 + F_i)| < B$ for $0 \leq p < q \leq n$, and

(2) $\Sigma_{i=1}^{n} |G_i| < B$.

Let g be the function defined on $[a, b]$ such that

(1) $g(a) = 1$, and

(2) $g(x) = 1 + \text{lub}\{\Sigma_J |G| : J$ a refinement of $\{x_i\}_{i=0}^{p-1} \cup \{x\}\}$, where $0 < p \leq n$ and $x_{p-1} < x \leq x_p$.

Thus, g is a positive nondecreasing function.

We use induction to establish the desired inequality. If $\{x_i\}_{i=0}^{n}$ is a refinement of D and $0 < p \leq q \leq n$, then

$$|S_{0pq}| = \left| \prod_{i=p}^{q} (1 + F_i) \right| \leq B.$$

Thus, the inequality is true for $j = 0$.

Suppose the inequality holds for the nonnegative integer j. That is, if $\{x_i\}_{i=0}^{n}$ is a refinement of D and $0 < p \leq q \leq n$, then $(\dagger)$ holds.

We now establish that the inequality also holds for $j + 1$. Suppose $\{x_i\}_{i=0}^{n}$ is a refinement of D and $0 < p \leq q \leq n$. To simplify notation in the following manipulations, let

$$f(v) = g(x_q) - g(v)$$

for $x_p \leq v \leq x_q$. Now,

$$|S_{j+1,p,q}| = \left|\sum_{i=p}^{q}\left[\prod_{k=p}^{i-1}(1+F_k)\right]G_i S_{j,i+1,q}\right|$$

$$\leq B \sum_{i=p}^{q} |G_i|\,|S_{j,i+1,q}|$$

$$\leq B \sum_{i=p}^{q} \{g(x_i) - g(x_{i-1})\}\{B^{j+1}[g(x_q) - g(x_i)]^j/j!\}$$

$$\leq B\left[(R)\int_{x_{p-1}}^{x_q} dg\{B^{j+1}[g(x_q) - g(v)]^j/j!\}\right]$$

$$= [B^{j+2}/j!]\left[(R)\int_{x_{p-1}}^{x_q} (-df)f^j\right]$$

$$\leq [B^{j+2}/(j+1)!] \sum_{k=0}^{j} (LR) \int_{x_{p-1}}^{x_q} (-df)f^{j-k}f^k$$

$$= [B^{j+2}/(j+1)!][f^{j+1}(x_{p-1}) - f^{j+1}(x_q)] \qquad \text{(Lemma 2)}$$

$$= B^{j+2}[g(x_q) - g(x_{p-1})]^{j+1}/(j+1)!.$$

Thus, the inequality holds for $j + 1$. Hence, the inequality is valid for $j = 0, 1, 2, \ldots$. Therefore, Lemma 4 is established.

Lemma 5. *If F and G are functions from $R \times R$ to N, $F \in OP^\circ$ on $[a, b]$ and $G \in OB^\circ$ on $[a, b]$, then $F + G \in OP^\circ$ on $[a, b]$.*

Proof. This lemma follows as a corollary to Lemmas 3 and 4.

Lemma 5 is established in a previous paper by the author for functions from $R \times R$ to R [5, Theorem 1 (1 → 2), p. 378]. However, the proof presented there is different from the proof employed in this paper.

Lemma 6. *If $\{F_i\}_{i=m}^n$ and $\{G_i\}_{i=m}^n$ are sequences of elements of N, then*

$$\prod_{i=m}^{n}(1 + F_i + G_i) = \prod_{i=m}^{n}(1 + F_i) + \sum_{i=m}^{n}\left[\prod_{j=m}^{i-1}(1+F_j)\right]G_i\left[\prod_{j=i+1}^{n}(1 + F_j + G_j)\right].$$

Proof. This lemma can be established by induction.

Lemma 7. *If G is a function from $R \times R$ to N and $G \in OB^\circ$ on $[a, b]$, then the following statements are equivalent:*

(1) $\int_a^b G$ *exists, and*

(2) ${}_x\Pi^y(1+G)$ *exists for* $a \leq x < y \leq b$.

Further, if $G \in OB^\circ$ *on* $[a, b]$ *and* (1) *or* (2) *is true, then* $G \in OM^*$ *on* $[a, b]$.

Proof. If $G \in OB^\circ$ on $[a, b]$ and ${}_x\Pi^y(1+G)$ exists for $a \leq x < y \leq b$, then $G \in OM^*$ on $[a, b]$ [7, Theorem 1]. Also, if $G \in OB^\circ$ on $[a, b]$, then $\int_a^b G$ exists if and only if ${}_x\Pi^y(1+G)$ exists for $a \leq x < y \leq b$ [7, Theorem 4]. Thus, the lemma follows.

We now establish Theorem 1.

Proof of Theorem 1 [(2), (3) $\to$ (1)]. We initially establish that $\Sigma_{i=0}^\infty P_i(x, y)$ converges uniformly and absolutely for $a \leq x < y \leq b$, where

$$P_0(x, y) = {}_x\prod^y(1+F)$$

and

$$P_i(x, y) = (LR)\int_x^y {}_x\prod^u(1+F)GP_{i-1}(v, y)$$

for $a \leq x < y \leq b$ and $i = 1, 2, \ldots$. The existence of these integrals follows by applying Lemma 1.

From Lemma 4, there exist a subdivision D_1 of $[a, b]$, a number B and a positive nondecreasing function g defined on $[a, b]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D_1, i is a nonnegative integer and $0 < p \leq q \leq n$, then

$$|S_{ipq}| \leq B^{i+1}[g(x_q) - g(x_{p-1})]^i/i!$$

and

$$|-S_{ipq}| \leq B^{i+1}[g(x_q) - g(x_{p-1})]^i/i!,$$

where S_{ipq} is defined as in Lemma 3.

It follows from the result stated in the preceding paragraph that

$$|P_i(x, y)| \leq B^{i+1}[g(y) - g(x)]^i/i!$$

for $a \leq x < y \leq b$ and $i = 0, 1, 2, \ldots$. Therefore, $\Sigma_{i=0}^\infty P_i$ converges uniformly and absolutely on $[a, b]$.

Suppose $a \leq x < y \leq b$. We now establish that ${}_x\Pi^y(1+F+G)$ exists and is $\Sigma_{i=0}^\infty P_i(x, y)$. Let $\epsilon > 0$.

There exists a positive integer N such that

$$\sum_{i=N+1}^{\infty} B^{i+1}[g(b) - g(a)]^i/i! < \epsilon/3.$$

Further, from the existence properties of the integrals involved, there exists a subdivision D_2 of $[a, b]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D_2 and $0 \leq p < q \leq n$, then

$$\left|\sum_{i=0}^{N} P_i(x_p, x_q) - \sum_{i=0}^{N} S_{i,p+1,q}\right| < \frac{\epsilon}{3}.$$

Let D denote a subdivision of $[x, y]$ which refines the intersection of $[x, y]$ and $D_1 \cup D_2$ and has at least $N + 1$ elements. Suppose $\{x_i\}_{i=0}^n$ is a refinement of D. Now,

$$\left|\sum_{i=0}^{\infty} P_i(x, y) - \prod_{i=1}^{n}(1 + F_i + G_i)\right|$$

$$= \left|\sum_{i=0}^{\infty} P_i(x, y) - \sum_{i=0}^{n} S_{i1n}\right| \qquad \text{(Lemma 3)}$$

$$\leq \left|\sum_{i=0}^{N} P_i(x, y) - \sum_{i=0}^{N} S_{i1n}\right| + \left|\sum_{i=N+1}^{\infty} P_i(x, y)\right| + \left|-\sum_{i=N+1}^{n} S_{i1n}\right|$$

$$\leq \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$

Hence, ${}_x\Pi^y(1 + F + G)$ exists and is $\Sigma_{i=0}^{\infty} P_i(x, y)$.

We now establish that $F + G \in OM^*$ on $[a, b]$. Since ${}_x\Pi^y(1 + F + G)$ exists for $a \leq x < y \leq b$, it is only necessary to establish the approximation part of the definition. Let $\epsilon > 0$. Further, let D_1, D_2 and N be defined as before.

Since F is in OM^*, OP° and S_2 on $[a, b]$ and G is in OB° and S_2 on $[a, b]$, there exists a subdivision D_3 of $[a, b]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of $[a, b]$, $0 \leq p < q \leq n$ and $q - p \leq N$, then

$$\left|{}_{x_p}\prod^{x_q}(1 + F + G) - \prod_{i=p+1}^{q}(1 + F_i + G_i)\right| < \epsilon.$$

Let D denote the subdivision $D_1 \cup D_2 \cup D_3$ of $[a, b]$. Suppose $\{x_i\}_{i=0}^n$ is a refinement of D and $0 \leq p < q \leq n$. If $q - p \leq N$, then the desired inequality follows immediately from the definition of D_3. If $q - p \geq N$, then

$$\left|{}_{x_p}\prod^{x_q}(1 + F + G) - \prod_{i=p+1}^{q}(1 + F_i + G_i)\right|$$

$$= \left|\sum_{i=0}^{\infty} P_i(x_p, x_q) - \sum_{i=0}^{p-q} S_{i,p+1,q}\right| \qquad \text{(Lemma 3)}$$

$$\leq \left|\sum_{i=0}^{N} P_i(x_p, x_q) - \sum_{i=0}^{N} S_{i,p+1,q}\right|$$

$$+ \left|\sum_{i=N+1}^{\infty} P_i(x_p, x_q)\right| + \left|-\sum_{i=N+1}^{p-q} S_{i,p+1,q}\right|$$

$$< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$

Therefore, $F + G \in OM^*$ on $[a, b]$. Thus, (2) and (3) imply (1).

Proof of Theorem 1 [(1), (2) → (3)]. Since F and $F + G$ are in OM^* and OP° on $[a, b]$, the existence of

$$(LR) \int_x^y {}_x\prod^u (1 + F) G \; {}_v\prod^y (1 + F + G)$$

for $a \leq x < y \leq b$ can be established by employing Lemma 6.

Since F and $F + G$ are in S_1 on $[a, b]$, there exists a subdivision $\{x_i\}_{i=0}^n$ of $[a, b]$ such that, if $1 \leq i \leq n$ and $x_{i-1} < x < y < x_i$, then

$$\left| {}_x\prod^y (1 + F) \right| < \frac{1}{2} \quad \text{and} \quad \left| {}_x\prod^y (1 + F + G) \right| < \frac{1}{2}.$$

Suppose $1 \leq i \leq n$ and $x_{i-1} < x < y < x_i$. Let J and K represent interval functions such that, if $x \leq u < v \leq y$, then

$$J(u, v) = {}_x\prod^u (1 + F) \quad \text{and} \quad K(u, v) = {}_v\prod^y (1 + F + G).$$

Since J and K are in OL° on $[a, b]$, it follows that J^{-1} and K^{-1} are also in OL° on $[a, b]$. Thus, from Lemma 1 and the existence of the integral in the preceding paragraph, we have that $\int_x^y G$ exists.

We have now established that, if $1 \leq i \leq n$ and $x_{i-1} < x < y < x_i$, then $\int_x^y G$ exists. From this and the fact that G is in OB° and S_2 on $[a, b]$, it follows that $\int_a^b G$ exists. Hence, $G \in OM^*$ on $[a, b]$ by Lemma 7. Thus, (1) and (2) imply (3).

Proof of Theorem 1 [(1), (3) → (2)]. It follows from Lemma 5 that $F + G \in OP^\circ$ on $[a, b]$. Further, $-G \in OM^*$ by Lemma 7. We have already established that (2) and (3) imply (1). Now, since $F + G - G \equiv F$, it follows that $F \in OM^*$ on $[a, b]$. Thus, (1) and (3) imply (2).

The proof of Theorem 1 is now complete. We next establish Theorem 2. One additional lemma is needed.

Lemma 8. *If G is a function from $R \times R$ to N and $G \in OB^\circ$ on $[a, b]$, then the following statements are equivalent*:

(1) $G \in OA^\circ$ *on* $[a, b]$, *and*

(2) $G \in OM^\circ$ *on* $[a, b]$ [2, *Theorem* 3.4, *p.* 301].

Proof of Theorem 2 [(2), (3) → (1)]. Since F and G are in OP° and OM° on $[a, b]$, F and G are also in OM^* on $[a, b]$. Hence, it follows from Theorem 1 [(2), (3) → (1)] that ${}_x\Pi^y(1 + F + G)$ exists for $a \leq x < y \leq b$. Thus, it is only necessary to show that $\int_a^b |1 + F + G - \Pi(1 + F + G)|$ exists and is zero in order to establish that $F + G \in OM^\circ$ on $[a, b]$. Let $\epsilon > 0$.

Since $F \in OM^\circ$ on $[a, b]$, there exists a subdivision D_1 of $[a, b]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D_1, then

$$\sum_{i=1}^{n}\left|1+F_i-{}_{x_{i-1}}\prod^{x_i}(1+F)\right|<\frac{\epsilon}{2}.$$

We know that F is in OP° and OM^* on $[a, b]$. Further, $F+G\in OP^\circ$ on $[a, b]$ by Lemma 5 and $F+G\in OM^*$ on $[a, b]$ by Theorem 1 [(2), (3) $\rightarrow$ (1)]. Now, since $G\in OB^\circ$ on $[a, b]$, it follows by using Lemma 6 that

$$(LR)\int_x^y {}_x\prod^u(1+F)G\,{}_v\prod^y(1+F+G)$$

exists and is

$${}_x\prod^y(1+F+G)-{}_x\prod^y(1+F)$$

for $a\le x<y\le b$.

Since F and $F+G$ are in S_1 and OM^* on $[a, b]$, for each positive number β there exists a subdivision $\{x_i\}_{i=0}^n$ of $[a, b]$ such that, if $1\le i\le n$ and $x_{i-1}<x<y<x_i$, then

$$\left|1-{}_x\prod^y(1+F)\right|<\beta \quad\text{and}\quad \left|1-{}_x\prod^y(1+F+G)\right|<\beta.$$

By Lemma 8, $G\in OA^\circ$ on $[a, b]$. Further, F and $F+G$ are in OP° on $[a, b]$ and $G\in OB^\circ$ on $[a, b]$. From these facts, it follows that

$$\int_a^b\left|G(x, y)-(LR)\int_x^y {}_x\prod^u(1+F)G\,{}_v\prod^y(1+F+G)\right|$$

exists and is zero. Thus, there exists a subdivision D_2 of $[a, b]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D_2, then

$$\sum_{i=1}^{n}\left|G_i-(LR)\int_{x_{i-1}}^{x_i} {}_{x_{i-1}}\prod^u(1+F)G\,{}_v\prod^{x_i}(1+F+G)\right|<\frac{\epsilon}{2}.$$

Let D denote the subdivision $D_1\cup D_2$ of $[a, b]$. Suppose $\{x_i\}_{i=0}^n$ is a refinement of D. Now,

$$\begin{aligned}
&\sum_{i=1}^{n}\left|1+F_i+G_i-{}_{x_{i-1}}\prod^{x_i}(1+F+G)\right| \\
&\quad=\sum_{i=1}^{n}\left|1+F_i+G_i-\left[{}_{x_{i-1}}\prod^{x_i}(1+F)\right.\right. \\
&\qquad\qquad\left.\left.+(LR)\int_{x_{i-1}}^{x_i} {}_{x_{i-1}}\prod^u(1+F)G\,{}_v\prod^{x_i}(1+F+G)\right]\right| \\
&\quad\le\sum_{i=1}^{n}\left|1+F_i-{}_{x_{i-1}}\prod^{x_i}(1+F)\right| \\
&\qquad\qquad+\sum_{i=1}^{n}\left|G_i-(LR)\int_{x_{i-1}}^{x_i} {}_{x_{i-1}}\prod^u(1+F)G\,{}_v\prod^{x_i}(1+F+G)\right| \\
&\quad<\epsilon/2+\epsilon/2=\epsilon.
\end{aligned}$$

Therefore, $F + G \in OM^{\circ}$ on $[a, b]$. Thus, (2) and (3) imply (1).

Proof of Theorem 2 [(1), (2) → (3)]. The proof of Theorem 2 [(1), (2) → (3)] is similar to the proof of Theorem 1 [(1), (2) → (3)]. The only difference is that it is necessary to use Lemma 8 rather than Lemma 7. Thus, (1) and (2) imply (3).

Proof of Theorem 2 [(1), (3) → (2)]. The proof of Theorem 2 [(1), (3) → (2)] is similar to the proof of Theorem 1 [(1), (3) → (2)]. As before, the only difference is that it is necessary to use Lemma 8 rather than Lemma 7. Thus, (1) and (3) imply (2).

The proof of Theorem 2 is now complete.

BIBLIOGRAPHY

1. W. D. L. Appling, *Interval functions and real Hilbert spaces*, Rend. Circ. Mat. Palermo (2) 11 (1962), 154–156. MR 27 #4040.

2. B. W. Helton, *Integral equations and product integrals*, Pacific J. Math. 16 (1966), 297–322. MR 32 #6167.

3. ———, *A product integral representation for a Gronwall inequality*, Proc. Amer. Math. Soc. 23 (1969), 493–500. MR 40 #1562.

4. J. C. Helton, *An existence theorem for sum and product integrals*, Proc. Amer. Math. Soc. 39 (1973), 149–154.

5. ———, *Bounds for products of interval functions*, Pacific J. Math. 49 (1973), 377–389.

6. ———, *Mutual existence of product integrals*, Proc. Amer. Math. Soc. 42 (1974), 96–103.

7. ———, *Mutual existence of sum and product integrals*, Pacific J. Math. 56 (1975).

8. A. Kolmogoroff, *Untersuchungen über den Integralbegriff*, Math. Ann. 103 (1930), 654–696.

9. J. S. MacNerney, *Integral equations and semigroups*, Illinois J. Math. 7 (1963), 148–173. MR 26 #1726.

10. P. R. Masani, *Multiplicative Riemann integration in normed rings*, Trans. Amer. Math. Soc. 61 (1947), 147–192. MR 8, 321.

DEPARTMENT OF MATHEMATICS, ARIZONA STATE UNIVERSITY, TEMPE, ARIZONA 85281